S.-T. Yau College Student Mathematics Contests 2016

Oral Exam of Geometry and Topology

1.

(1) Let $F : S^n \to S^n$ be a continuous map.
Define the degree of F and show that when F is smooth,

$$degF \int_{S^n} \omega = \int_{S^n} F^*\omega$$

for all $\omega \in \Omega^n(S^n)$.

(2) Show that if F has no fixed point then $degF = (-1)^{n+1}$.

2. Let $n \geq 0$ be an integer, M be a compact smooth manifold of dimension $4n + 2$, show that $dimH^{2n+1}(M, \mathbb{R})$ is even.

3. Let M be a compact, simply connected smooth manifold of dimension n, prove that there is no smooth immersion

$$f : M \to T^n,$$

where T^n is n-torus.

4. Let $a > 0$ be a real number.
Let $S'(a)$ denote the circle obtained by identifying the end points of the interral $[0, a]$.
Consider the Riemannian metric defined by

$$r^2\left(1 - \frac{1}{r^2}\right)dx^2 + \left(1 - \frac{1}{r^2}\right)^{-1} r^{-2}dr^2,$$

where $x \in S'(a)$ and $r \in (1, \infty)$.
Find the value of a such that the metric can be smoothly extended to $r = 1$.